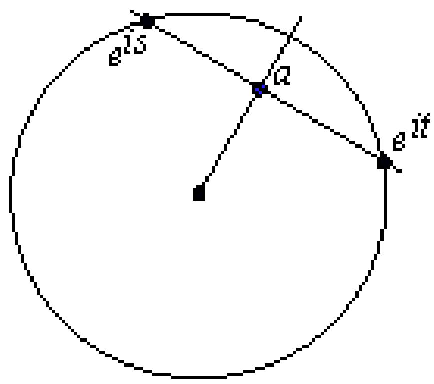
Answer to the Problem 4g) on the worksheet "Complex second degree polynomials" at
`http://wanda.uef.fi/mathematics/MathDistEdu/SemProd/ComplexSecDPol.htm`

We shall use the following apparent properties of complex numbers:

1 A complex number z has unit module if and only if there is a real number t such that $z = e^{it}$.

2 A complex number a can be expressed in a form $a = \frac{1}{2}(e^{is} + e^{it})$ with s, t real numbers, if and only if $|a| < 1$.
For the proof, see the figure below.



3 Recall the definition of a double point: The locus of a complex function f has a *double point* if there are two distinct points z and w such that $f(z) = f(w)$.

Proposition Let f be the complex function defined by $f(z) = z^2 + bz + c$, with zeros z_1 and z_2. If z is describing the circle $C(P, r)$ with center P and radius $r > 0$, then the locus (image) $\{\, f(z) \mid z \in C(P, r) \,\}$ has a double point if and only if the midpoint $m := \frac{1}{2}(z_1 + z_2)$ of the segment between the two zeros is inside the circle $C(P, r)$.
Proof. The circle is represented by $C(P, r) = \{\, P + re^{it} \mid t \in [0, 2\pi[\,\}$. Then the locus is described by $\gamma : [0, 2\pi[\to \mathbb{C}$,

$$\gamma(t) = f(P + re^{it}).$$

Note that

$$f(z) = (z - z_1)(z - z_2) = \Big(z - \frac{z_1 + z_2}{2}\Big)^2 - \Big(\frac{z_1 - z_2}{2}\Big)^2 = (z - m)^2 - \Big(\frac{z_1 - z_2}{2}\Big)^2$$

(show in details!). We are looking for $s \neq t$ such that $s, t \in [0, 2\pi[$ and

$$(P + re^{is} - m)^2 - \Big(\frac{z_1 - z_2}{2}\Big)^2 = (P + re^{it} - m)^2 - \Big(\frac{z_1 - z_2}{2}\Big)^2,$$

which is equivalent to (calculate!)

$$r^2(e^{is} - e^{it})\Big(e^{is} + e^{it} + 2\,\frac{k - m}{r}\Big) = 0.$$

Now we have $r \neq 0$, $e^{is} \neq e^{it}$, so that $e^{is} + e^{it} = 2\frac{m-k}{r}$. But by the fact 2 above this happens if and only if $|m - k| < r$. But this means that m is inside the circle.