Linear functions in the plane

A function $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a *linear function* or *linear transformation*, if

$$L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) + L\left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \quad \text{and} \quad L\left(c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = cL\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$$

Matrix and linear function in the plane We use the abbreviation $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function. Then we can calculate

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = L\left(x_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = x_1 \underbrace{L\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{=:\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}} + x_2 \underbrace{L\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{=:\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}} = x_1\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{=:A_L}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This means that the matrix A_L of the function L is the matrix formed from the images of the standard base of $\mathbb{R}^2$ and

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_L \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$

Matrix A_L eigenvalues and eigenvectors

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function with matrix $A := A_L$. Let us denote $\mathbf{x} = \binom{x_1}{x_2}$, so that $L(\mathbf{x}) = A\mathbf{x}$.
Then the *eigenvalue equation* of L, as well as of A, is the matrix-scalar equation $L(\mathbf{x}) = c\mathbf{x}$ or, with the matrix, $A\mathbf{x} = c\mathbf{x}$, where the scalar c and vector $\mathbf{x}$ are the unknowns.
Every scalar solution c is an *eigenvalue* of L and A, and for each eigenvalue c the corresponding solution vectors $\mathbf{x}$ form the *eigenspace*

$$E_c = \{\, \mathbf{x} \mid L(\mathbf{x}) = c\mathbf{x} \,\}$$

Nonzero eigenspace vectors are *eigenvectors* corresponding to the eigenvalue c.

Characteristic equation

We have $c\mathbf{x} = cI\mathbf{x}$, where I is the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Therefore

$$A\mathbf{x} = c\mathbf{x} \iff A\mathbf{x} - cI\mathbf{x} = \mathbf{0} \iff (A - cI)\mathbf{x} = \mathbf{0}.$$

The quadratic homogeneous equation $(A - cI)\mathbf{x} = \mathbf{0}$ has non-trivial solutions if and only if

$$\det(A - cI) = 0.$$