BOOLEAN TENSOR FACTORIZATIONS

Pauli Miettinen

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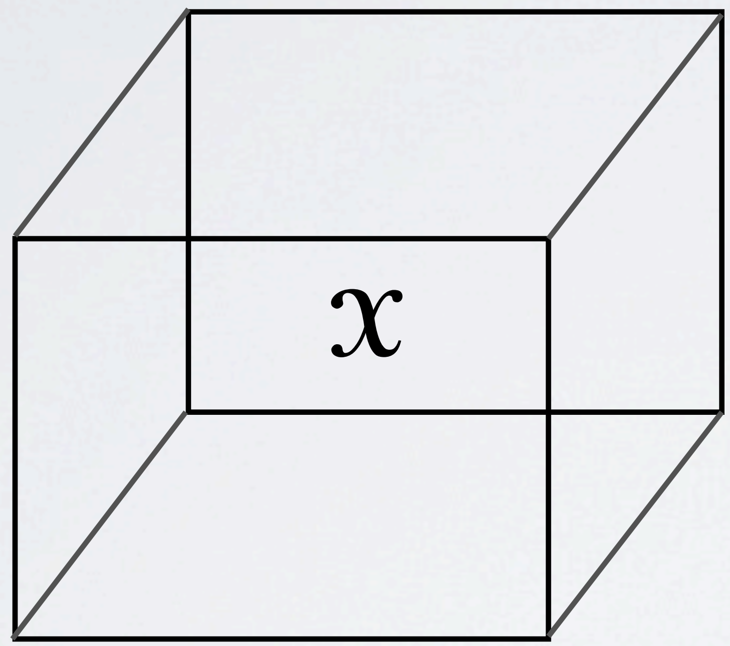
BACKGROUND: TENSORS AND TENSOR FACTORIZATIONS



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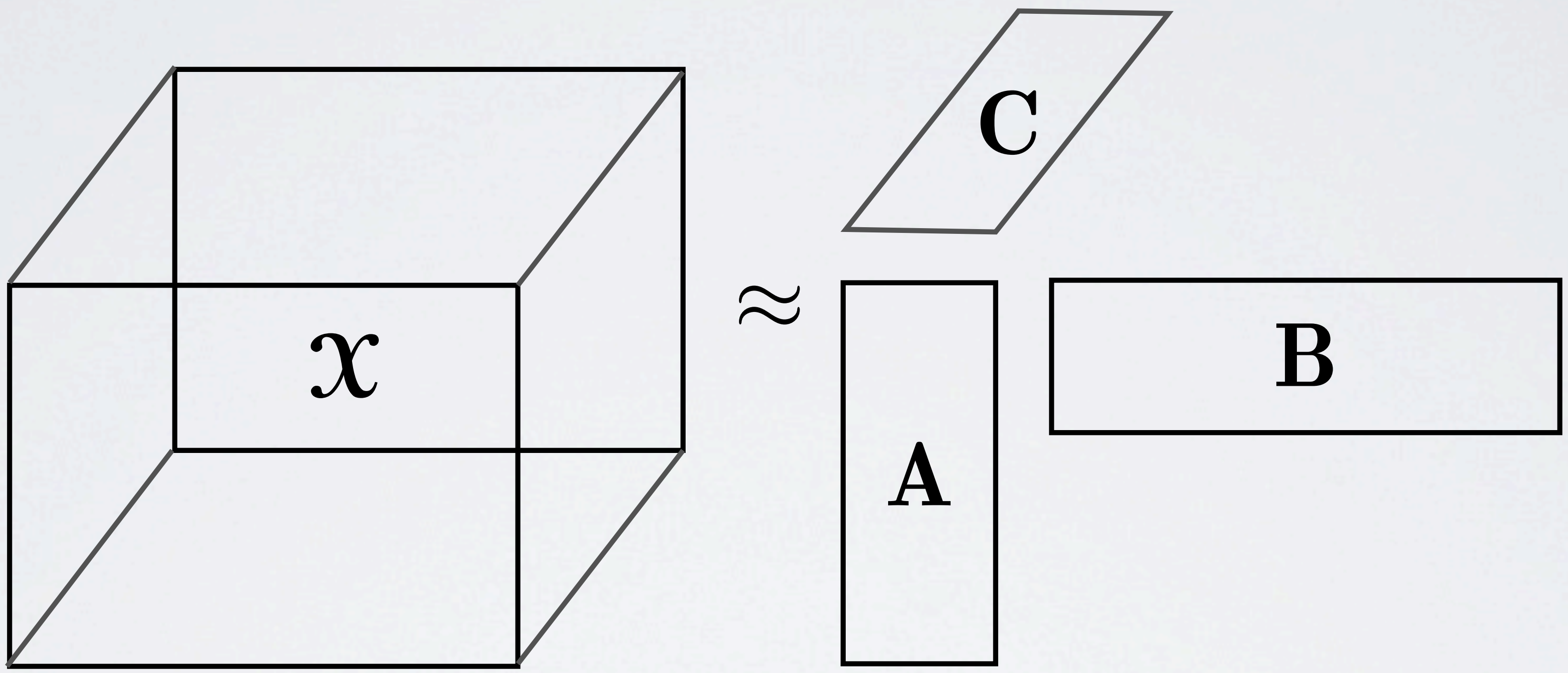

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BACKGROUND: BOOLEAN MATRIX FACTORIZATIONS

- Given a binary matrix **X** and a positive integer *R*, find two binary matrices **A** and **B** such that **A** has *R* columns and **B** has *R* rows and **X** ≈ **A** ∘ **B**.
  - **A** ∘ **B** is the **Boolean matrix product** of **A** and **B**,

$$(\mathbf{A} \circ \mathbf{B})_{ij} = \bigvee_{r=1}^{R} b_{il} c_{lj}$$

BOOLEAN TENSOR FACTORIZATIONS: THE IDEA

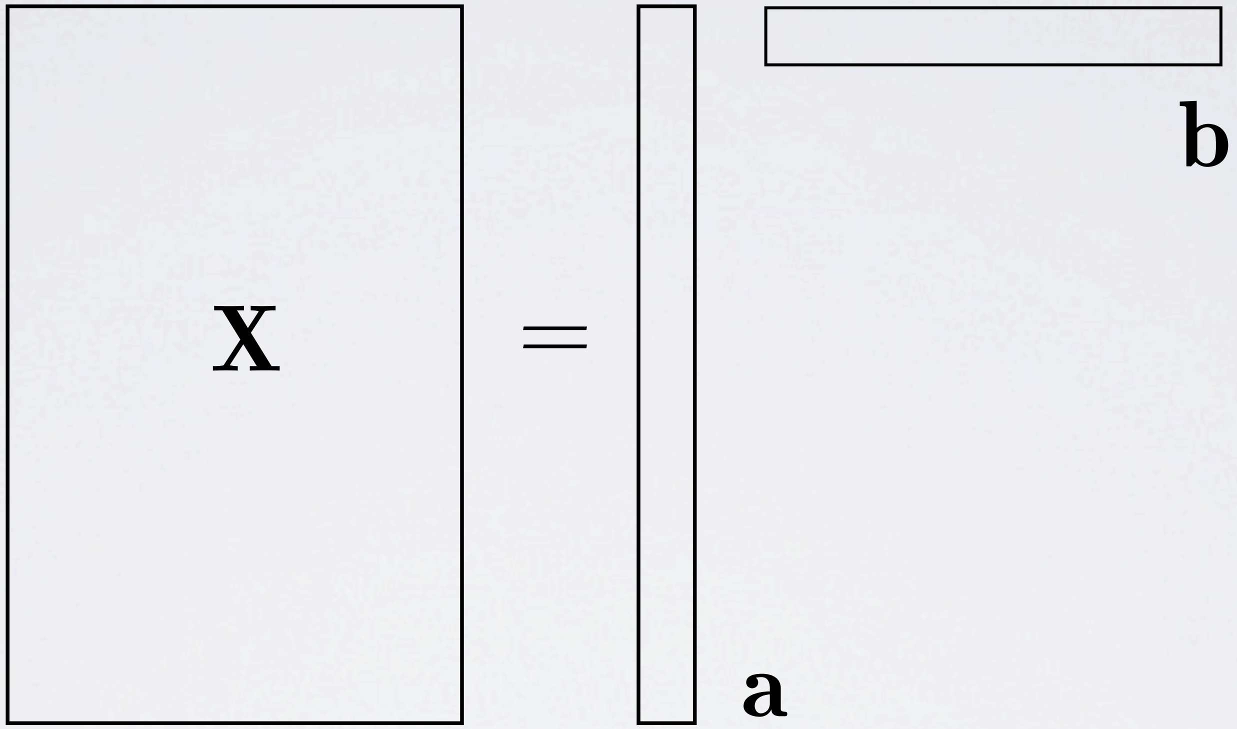
1. Take existing (normal) tensor factorization

2. Make everything binary and define summation as 1 + 1 = 1

3. Try to understand what you just did.

Research problem. What can we say about Boolean tensor factorizations and how do they relate to normal tensor factorizations and Boolean matrix factorizations?

RANK-1 (BOOLEAN) TENSORS



$$\mathbf{X} = \mathbf{a} \times \mathbf{b}$$

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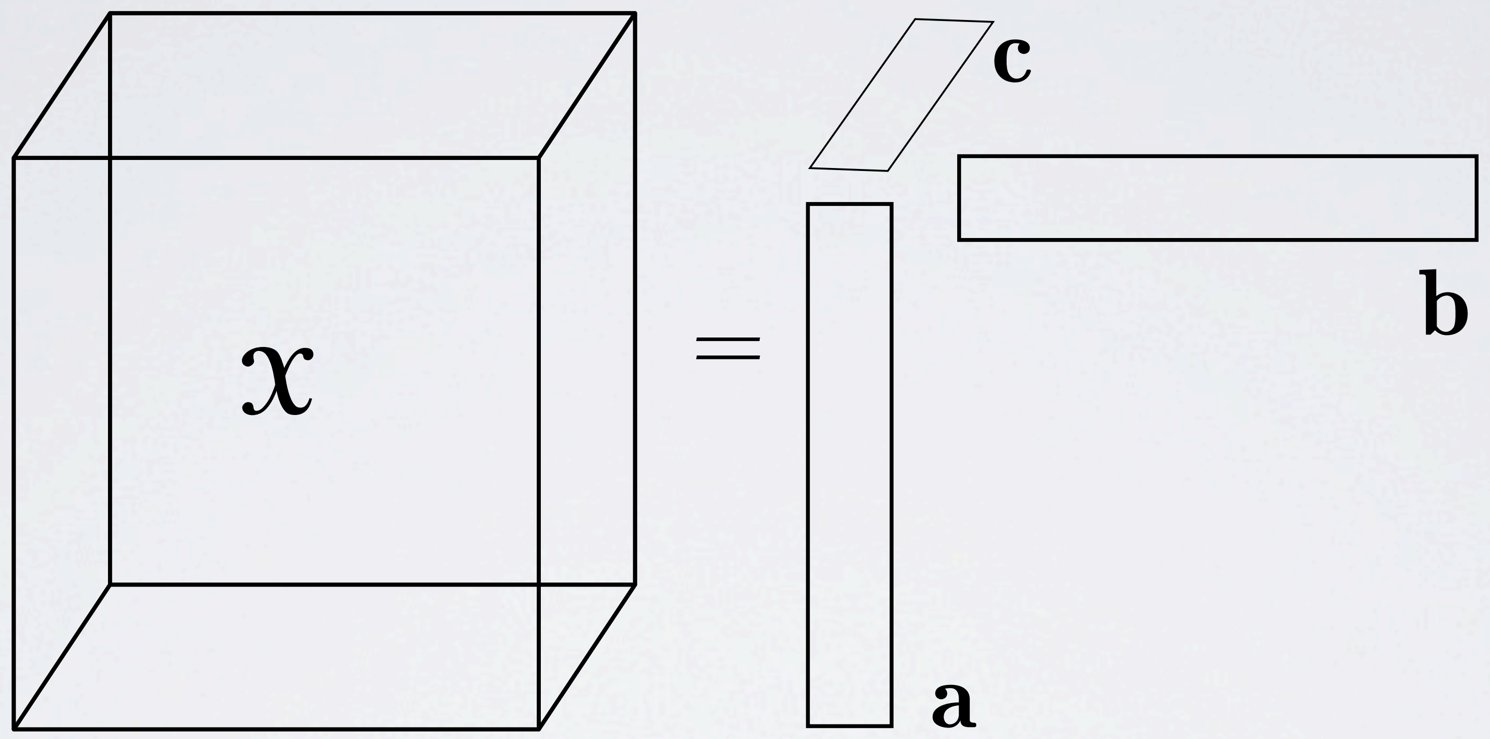
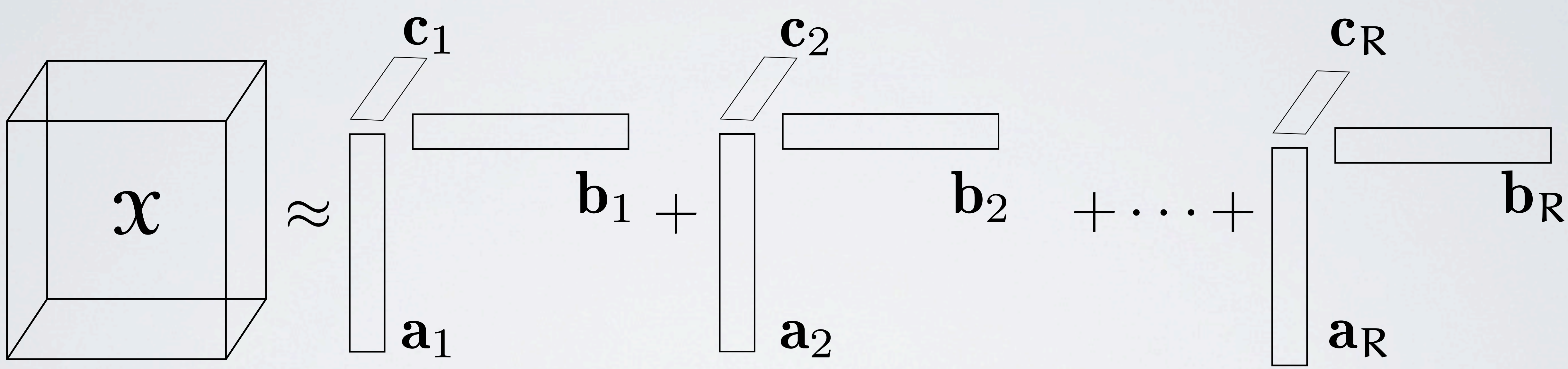


$$\mathcal{X} = \mathbf{a} \times_1 \mathbf{b} \times_2 \mathbf{c}$$

THE CP TENSOR DECOMPOSITION



$$x_{ijk} \approx \sum_{r=1}^{R} a_{ir} b_{jr} c_{kr}$$

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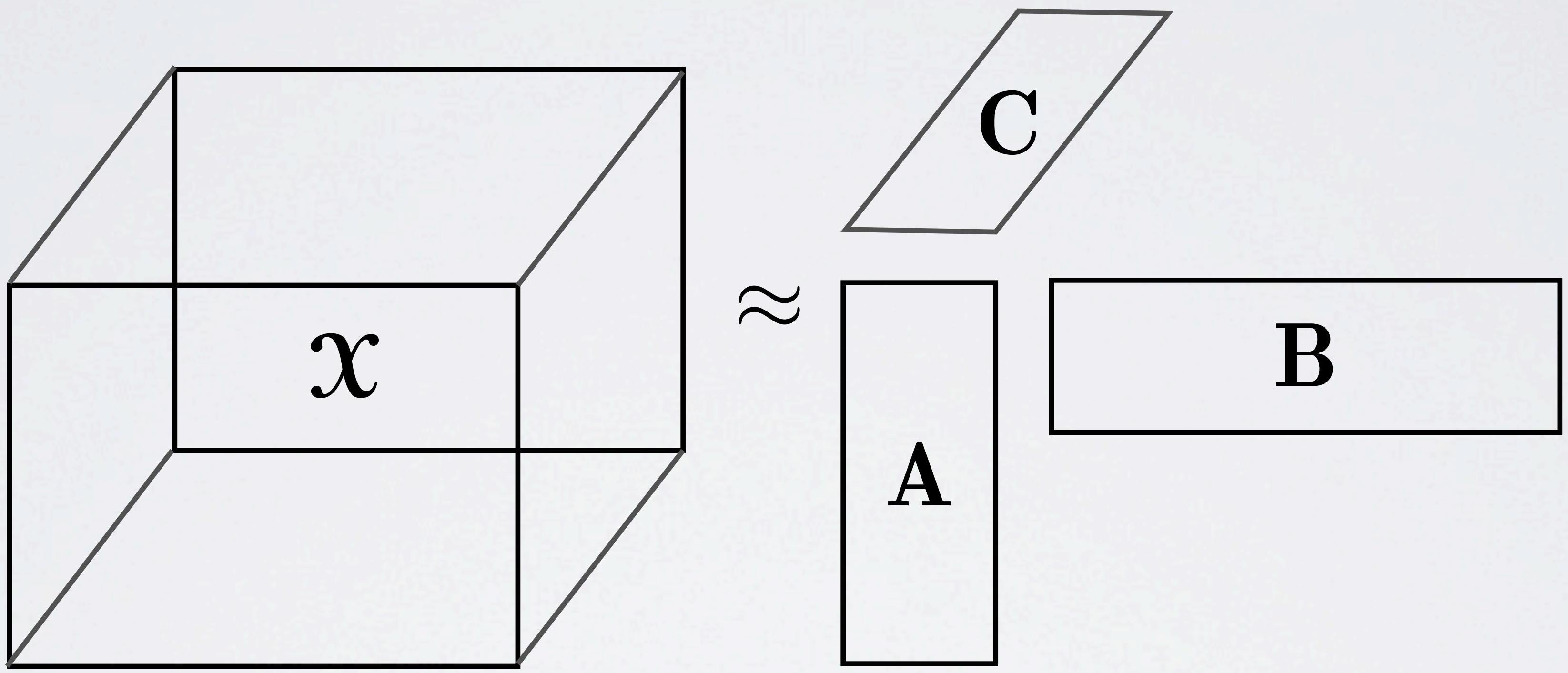

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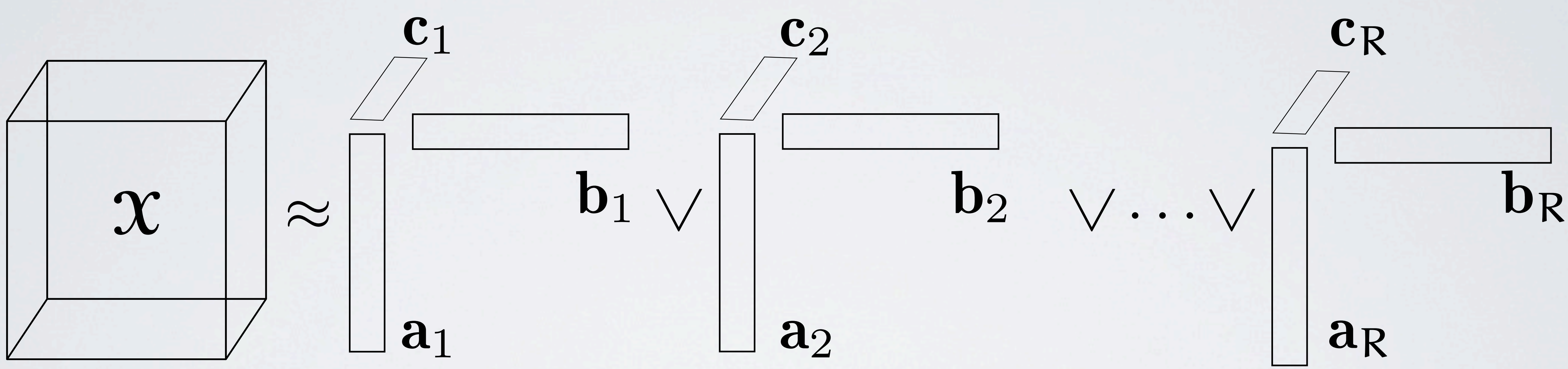

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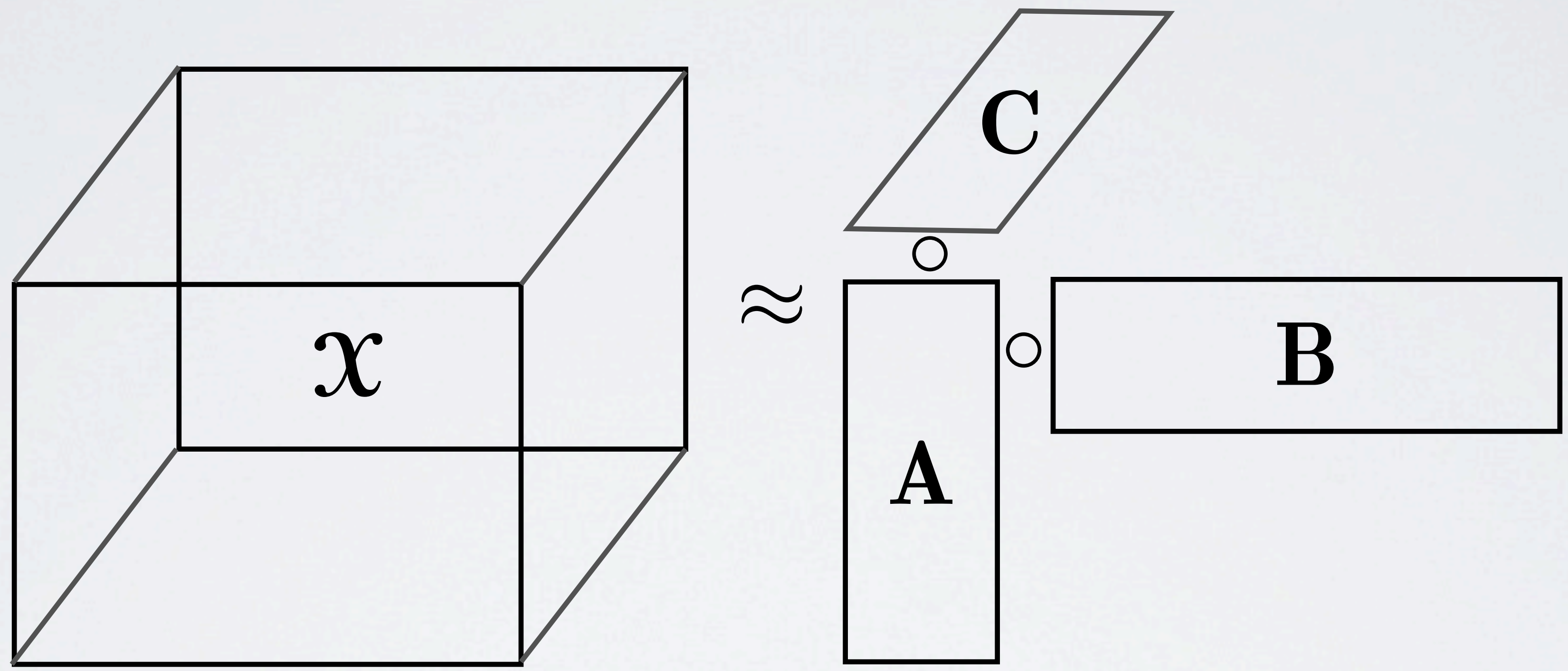

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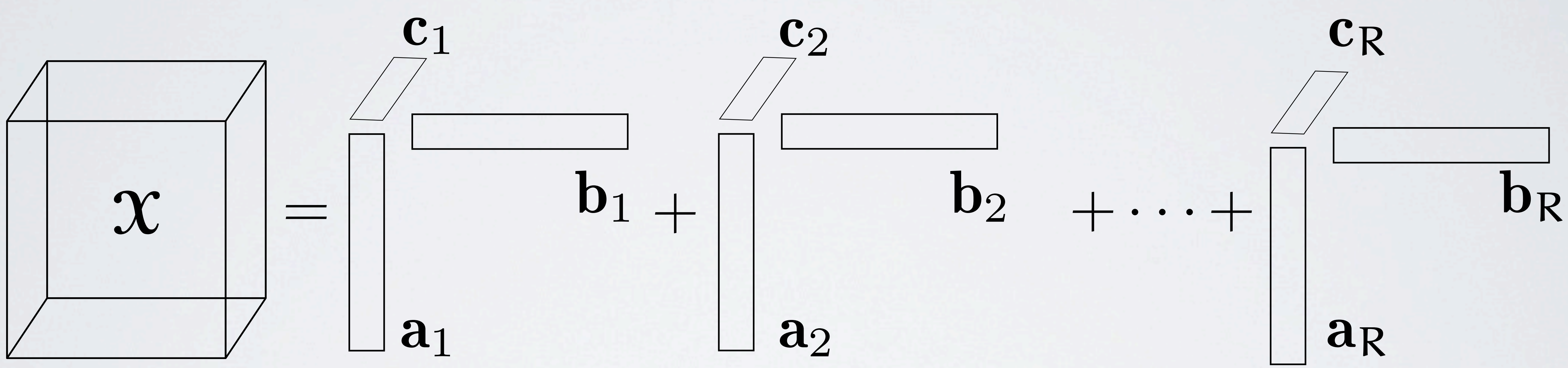
DIGRESSION: FREQUENT TRI-ITEMSET MINING

- Rank-1 *N*-way binary tensors define an *N*-way itemset
  - Particularly, rank-1 binary matrices define an itemset
  - In itemset mining the induced sub-tensor must be full of 1s
  - Here, the items can have holes
- Boolean CP decomposition = lossy *N*-way tiling

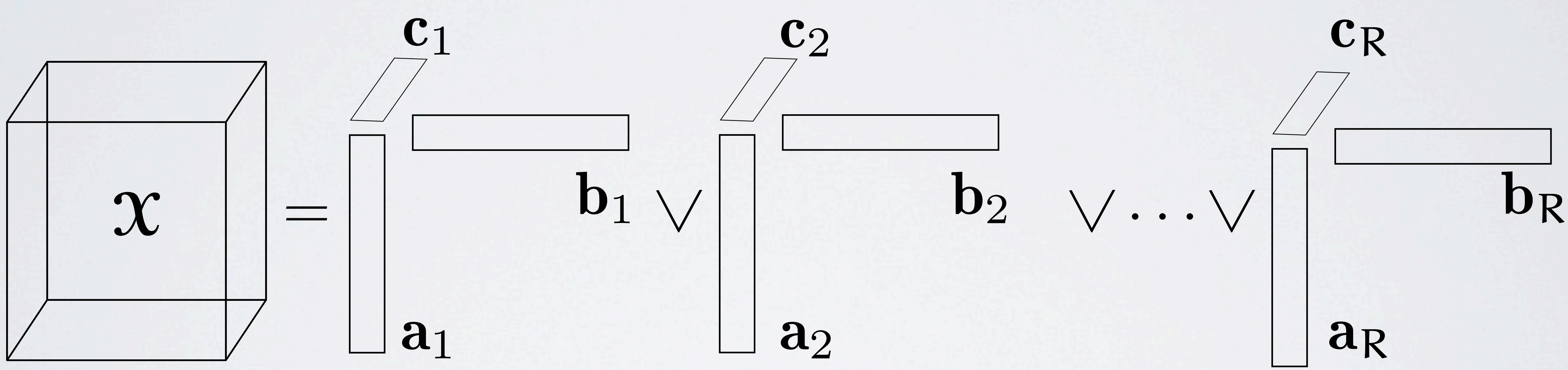
TENSOR RANK

The **rank** of a tensor is the minimum number of rank-1 tensors needed to represent the tensor exactly.



BOOLEAN TENSOR RANK

The **Boolean rank** of a binary tensor is the minimum number of binary rank-1 tensors needed to represent the tensor exactly using Boolean arithmetic.



SOME RESULTS ON RANKS

- Normal tensor rank is NP-hard to compute
- Normal tensor rank of n-by-m-by-k tensor can be more than $\min\{n, m, k\}$
  - But no more than $\min\{nm, nk, mk\}$

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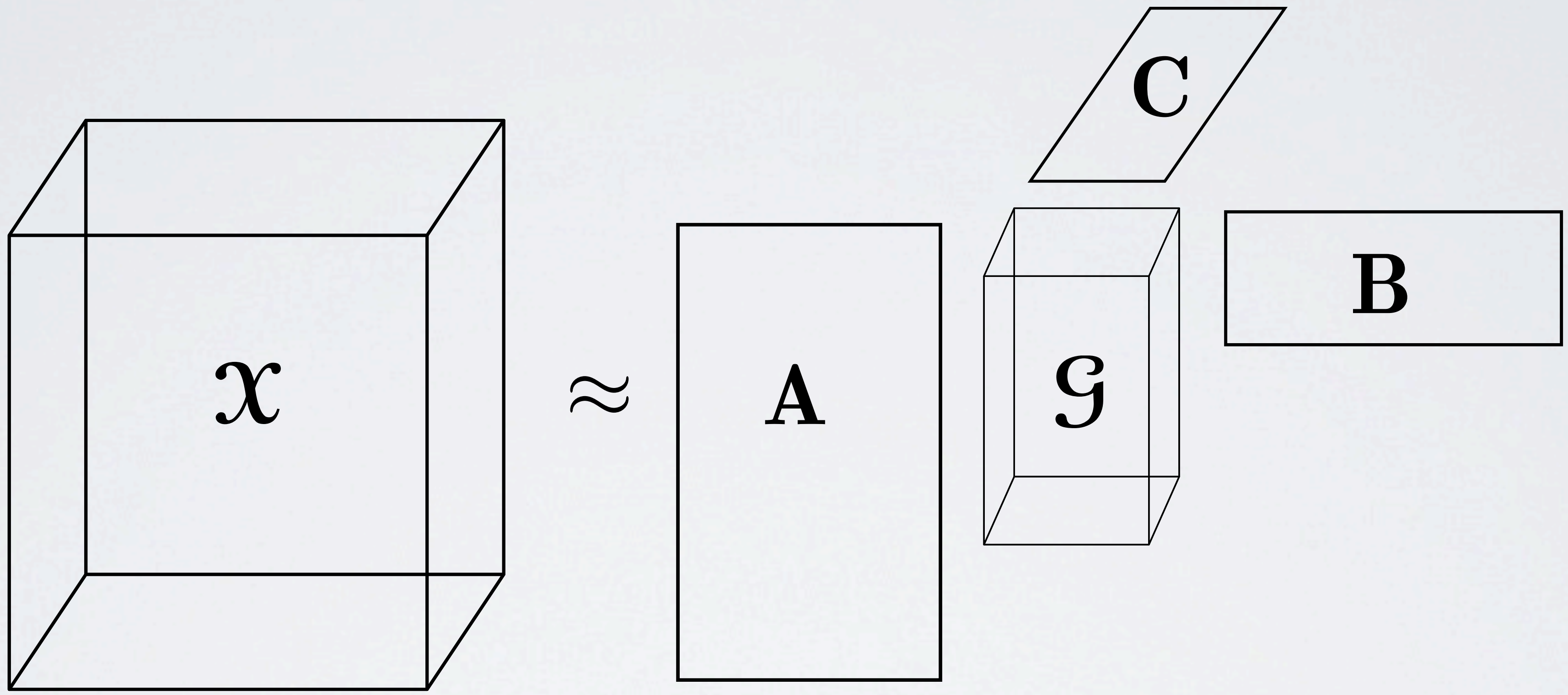
SPARSITY

- Binary matrix $\mathbf{X}$ of Boolean rank R and $|\mathbf{X}|$ 1s has Boolean rank-R decomposition $\mathbf{A} \circ \mathbf{B}$ such that $|\mathbf{A}| + |\mathbf{B}| \leq 2|\mathbf{X}|$ [M., ICDM '10]

- Binary N-way tensor $\mathcal{X}$ of Boolean tensor rank R has Boolean rank-R CP-decomposition with factor matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$ such that $\sum_i |\mathbf{A}_i| \leq N|\mathcal{X}|$

  - Both results are existential only and extend to approximate decompositions

THE TUCKER TENSOR DECOMPOSITION



$$x_{ijk} \approx \sum_{p=1}^{P} \sum_{q=1}^{Q} \sum_{r=1}^{R} g_{pqr} a_{ip} b_{jq} c_{kr}$$

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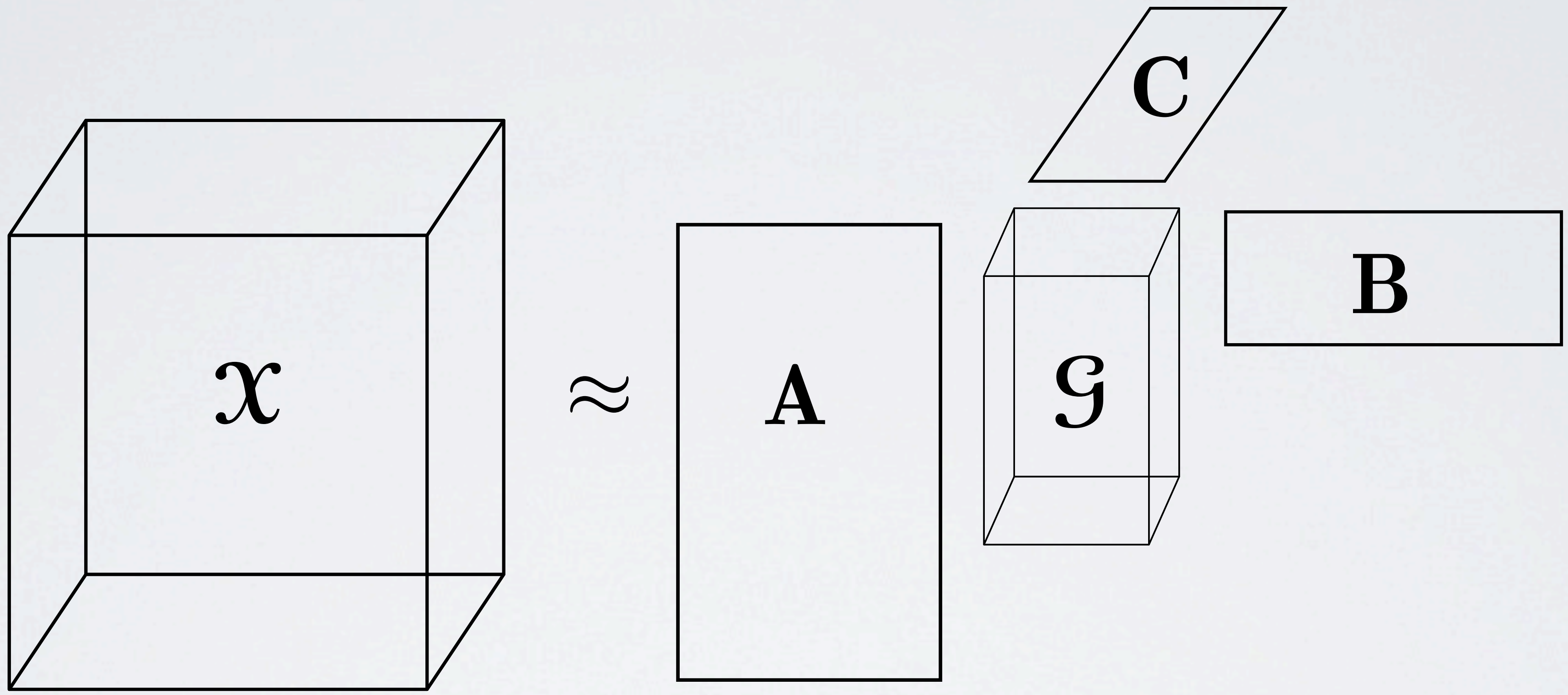

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THE ALGORITHMS

- The normal CP-decomposition can be solved using matricization and ALS
  - $\odot$ is the Khatri–Rao matrix product
  - $(\mathbf{C} \odot \mathbf{B})^T$ is R-by-mk
- For normal matrices, we can use standard least-squares projections
  - One projection per mode
- Similar algorithms for the Tucker decomposition

$$\mathbf{X}_{(1)} = \mathbf{A}(\mathbf{C} \odot \mathbf{B})^{\mathsf{T}}$$
$$\mathbf{X}_{(2)} = \mathbf{B}(\mathbf{C} \odot \mathbf{A})^{\mathsf{T}}$$
$$\mathbf{X}_{(3)} = \mathbf{C}(\mathbf{B} \odot \mathbf{A})^{\mathsf{T}}$$

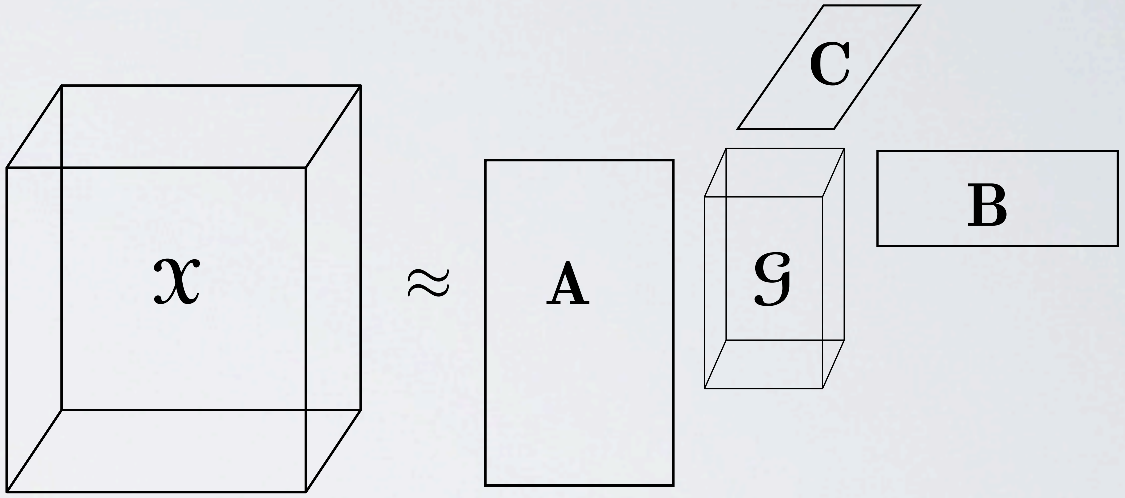
THE ALGORITHMS

- For Boolean case, matrix product must be changed
  - Khatri–Rao stays as it
  - Finding the optimal projection is NP-hard even to approximate
- Good initial values are needed due to multiple local minima
  - Obtained using Boolean matrix factorization to matricizations

$$\mathbf{X}_{(1)} = \mathbf{A} \circ (\mathbf{C} \odot \mathbf{B})^T$$
$$\mathbf{X}_{(2)} = \mathbf{B} \circ (\mathbf{C} \odot \mathbf{A})^T$$
$$\mathbf{X}_{(3)} = \mathbf{C} \circ (\mathbf{B} \odot \mathbf{A})^T$$

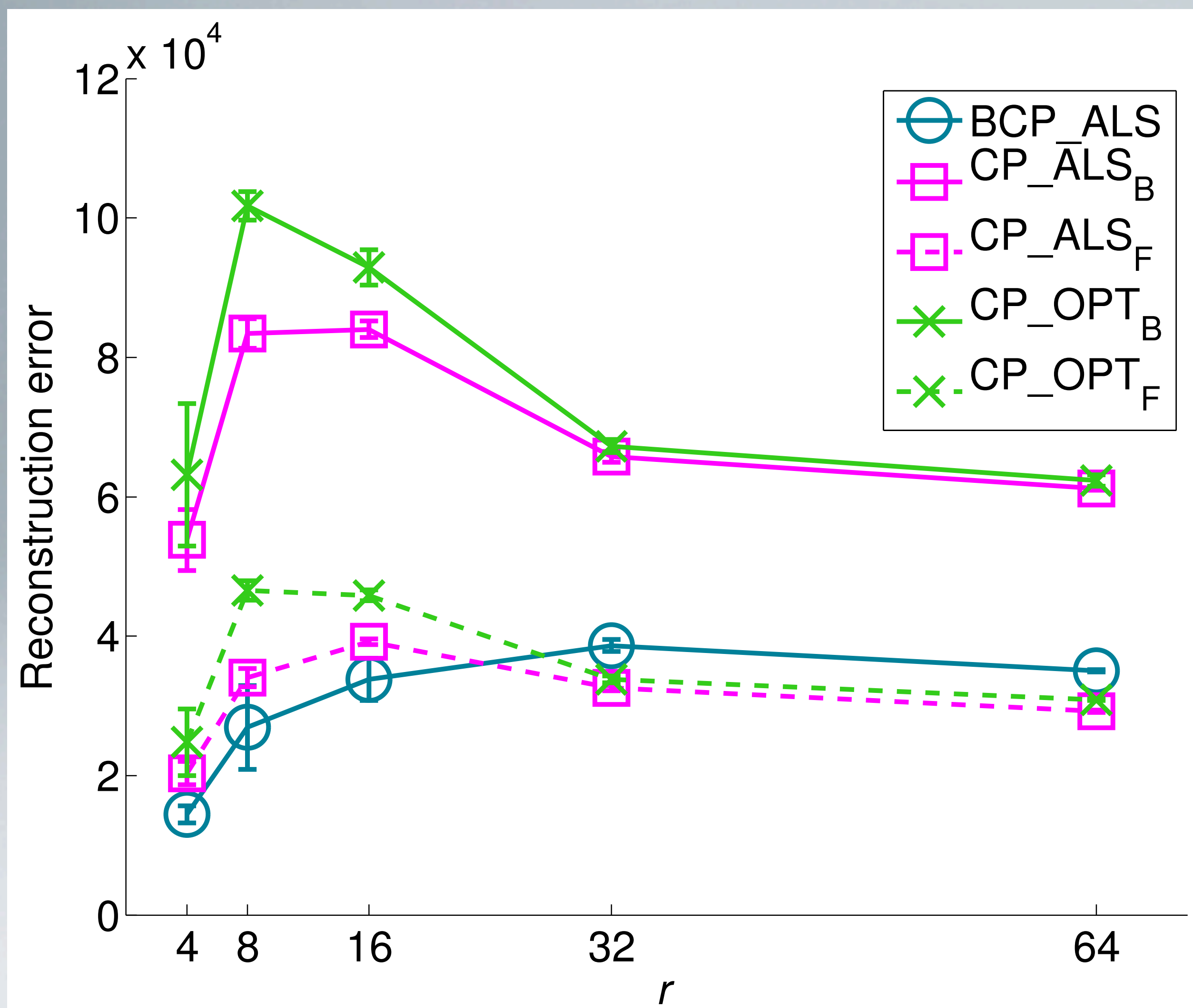
THE TUCKER CASE

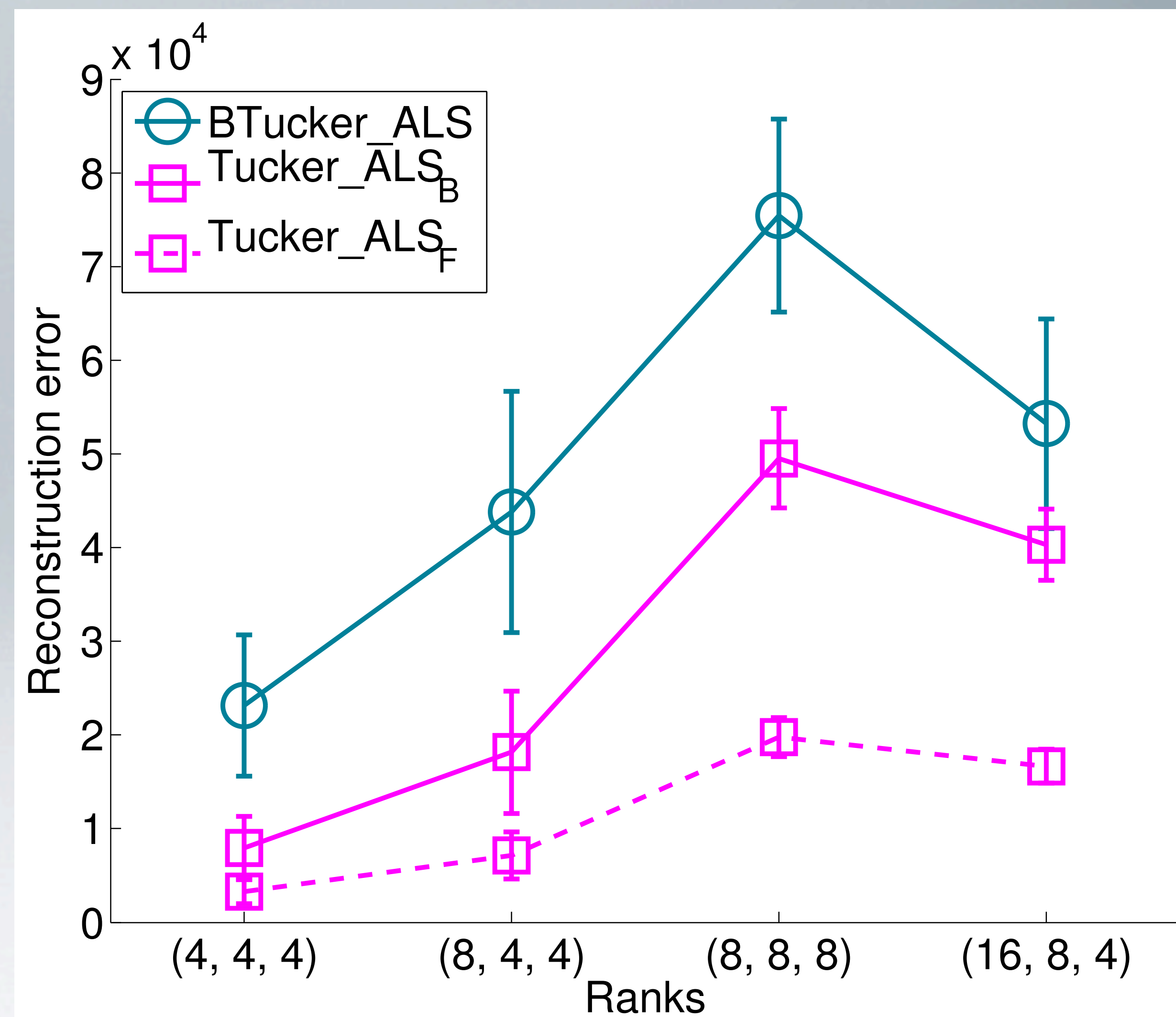
- The core tensor has global effects
  - Updates are hard
- Core tensor is usually small
  - We can afford more time per element
  - In Boolean case many changes make no difference



$$x_{ijk} \approx \bigvee_{p=1}^{P} \bigvee_{q=1}^{Q} \bigvee_{r=1}^{R} g_{pqr} a_{ip} b_{jq} c_{kr}$$

SYNTHETIC EXPERIMENTS





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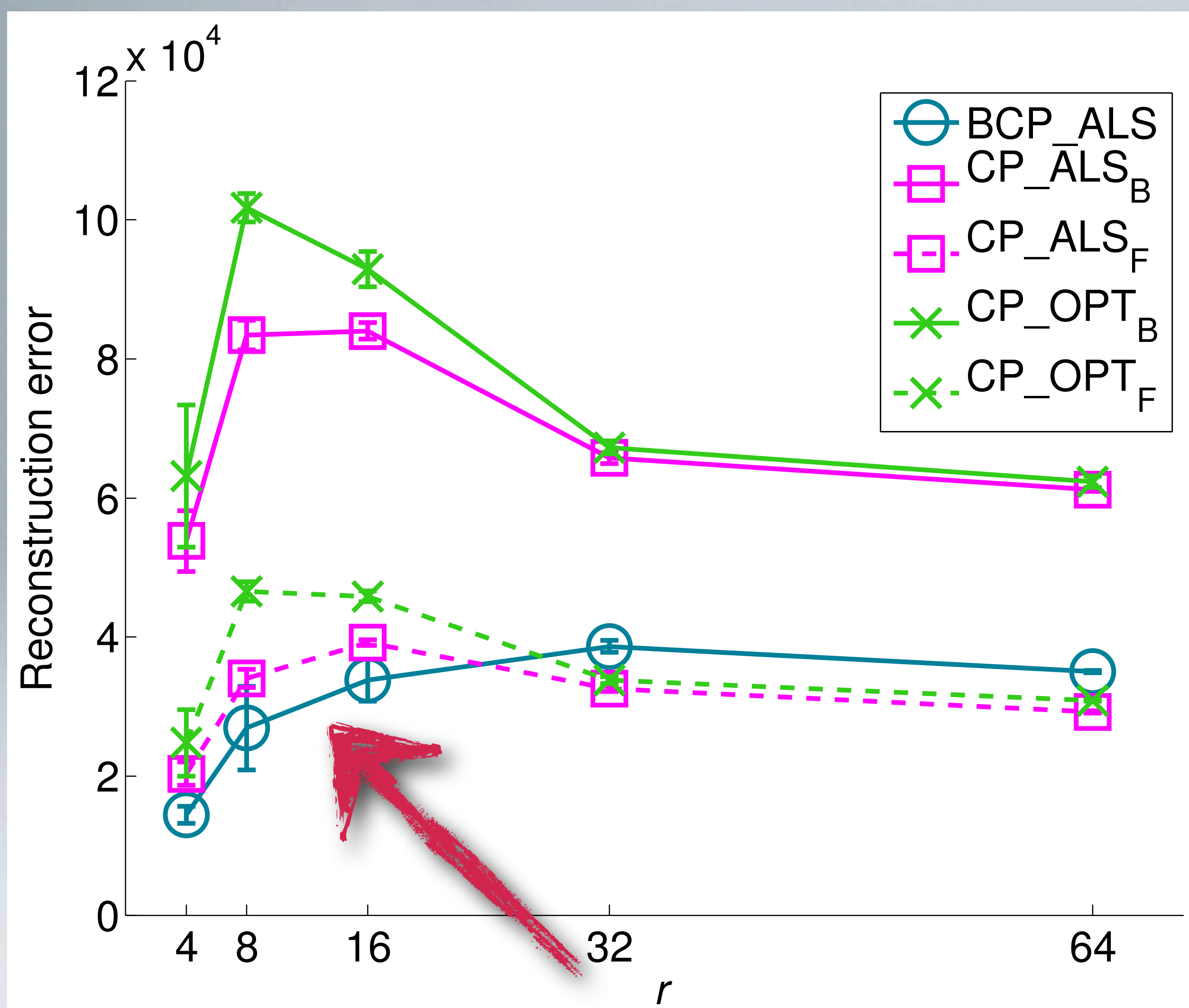

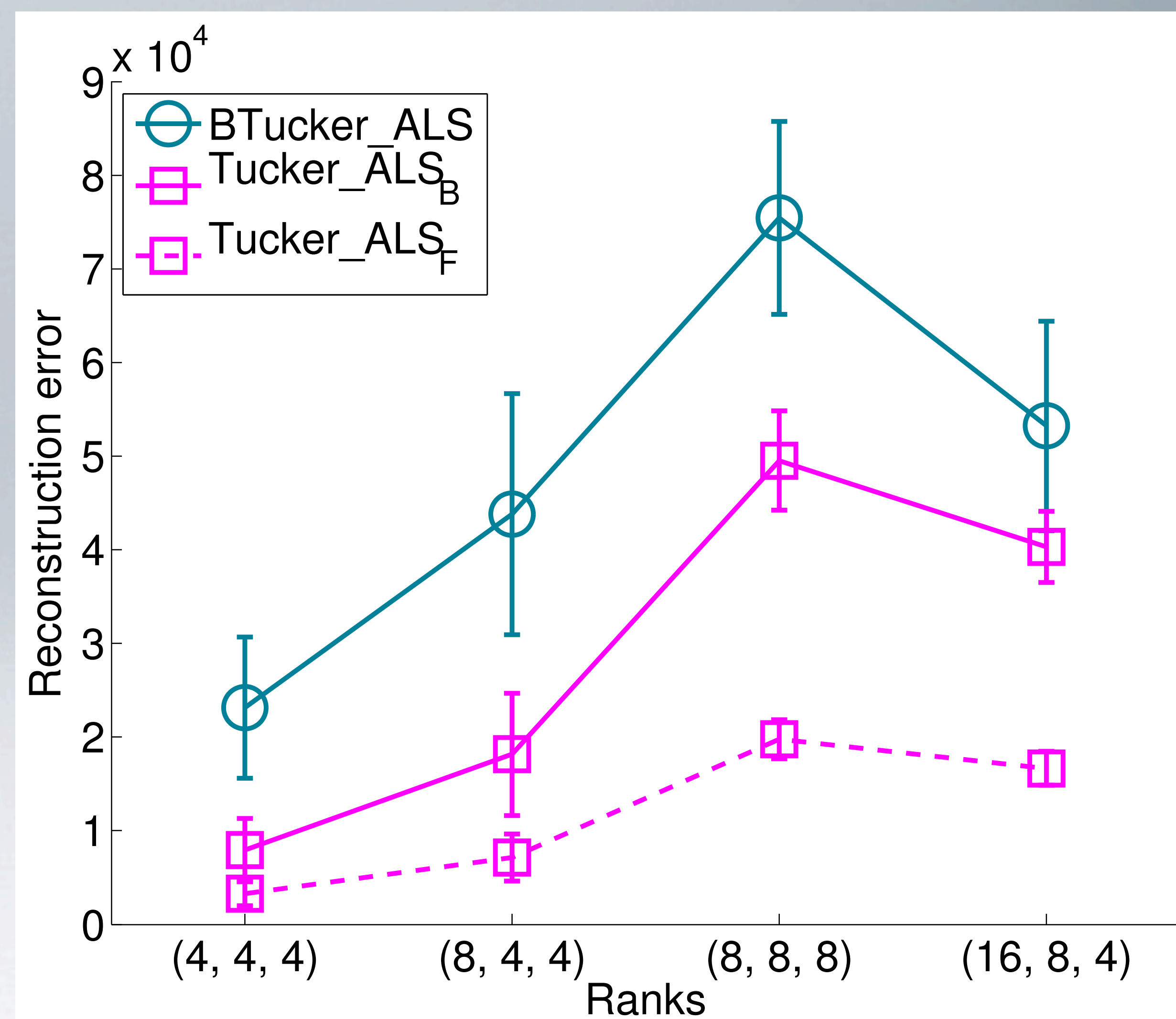

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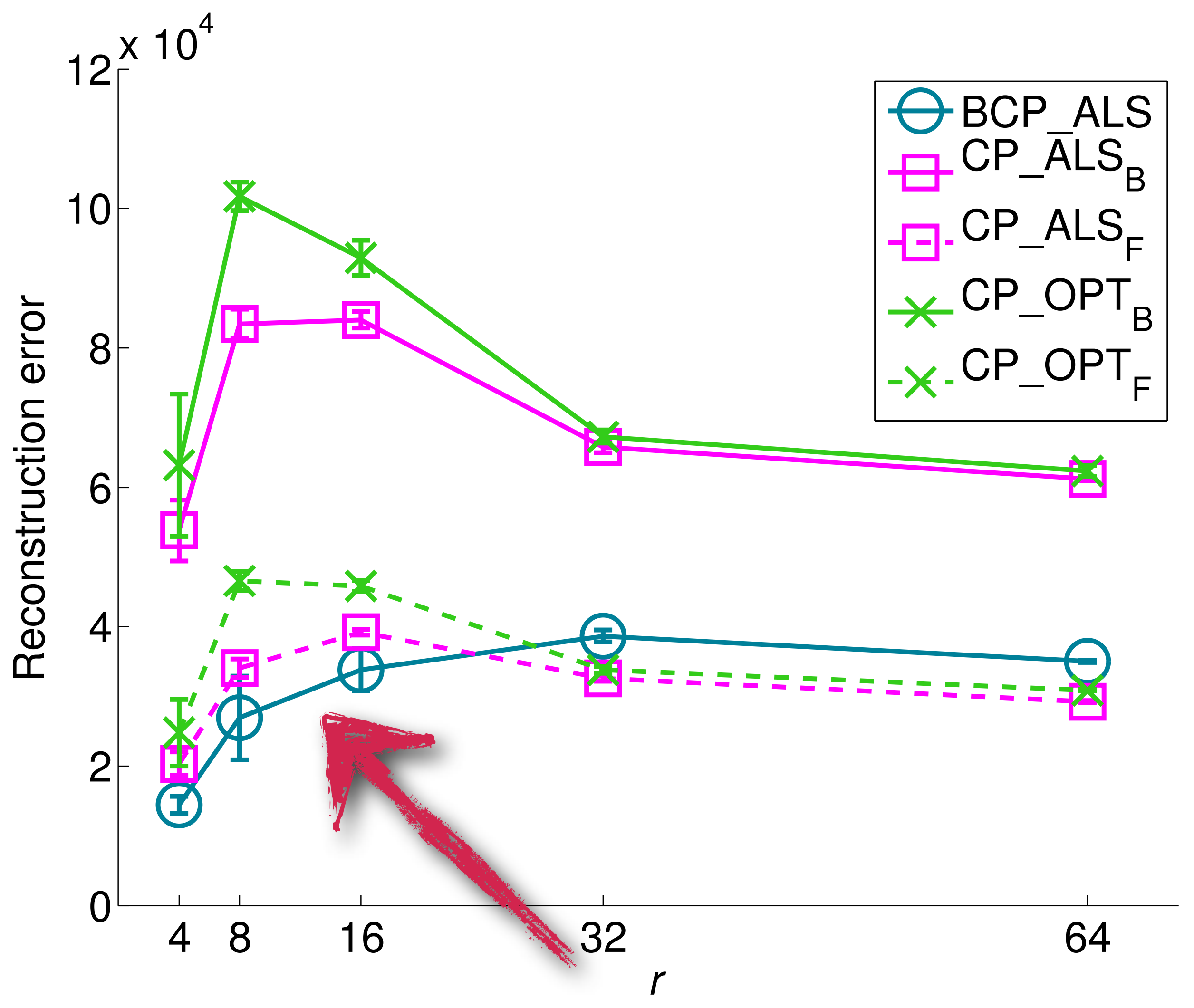
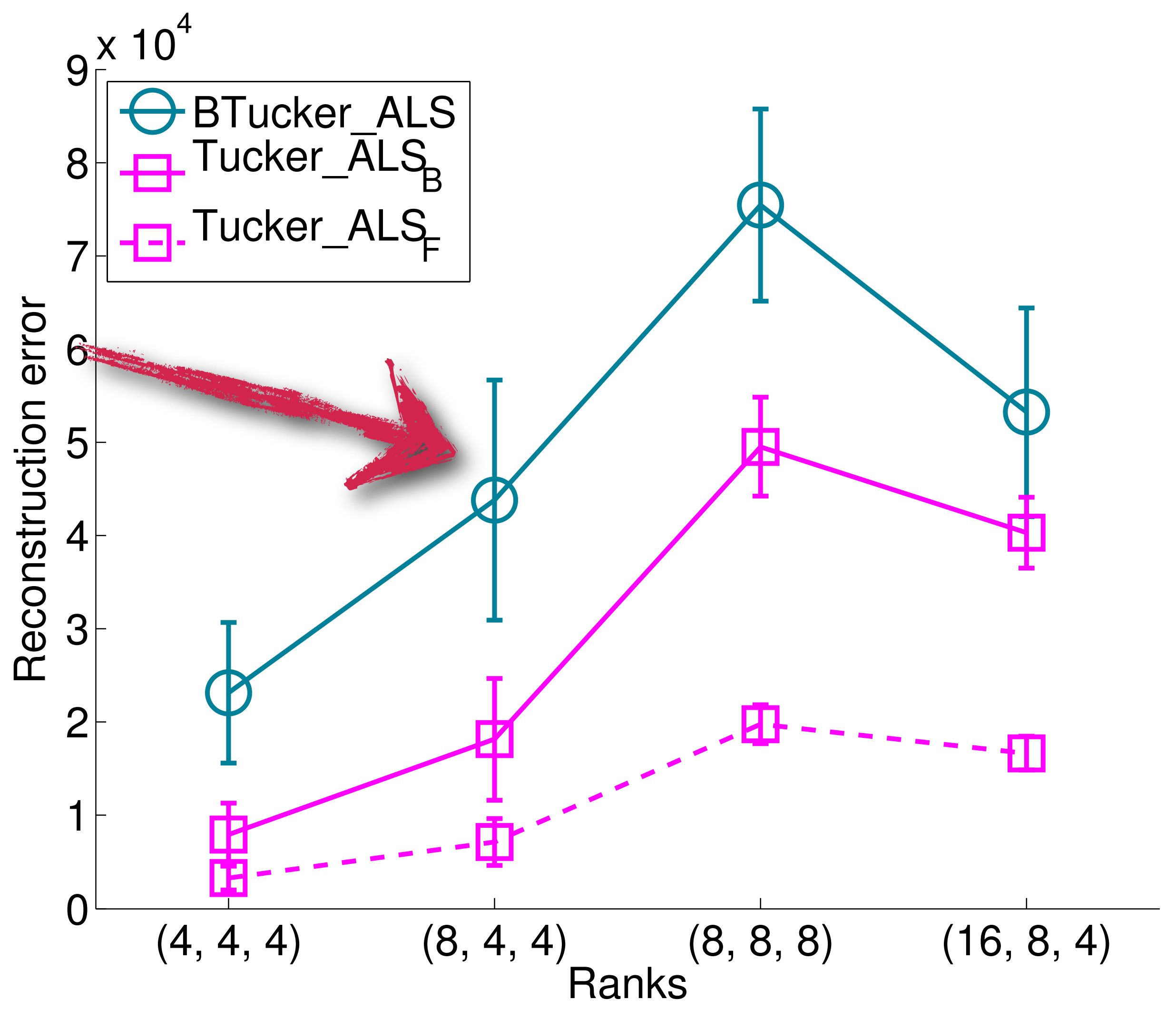
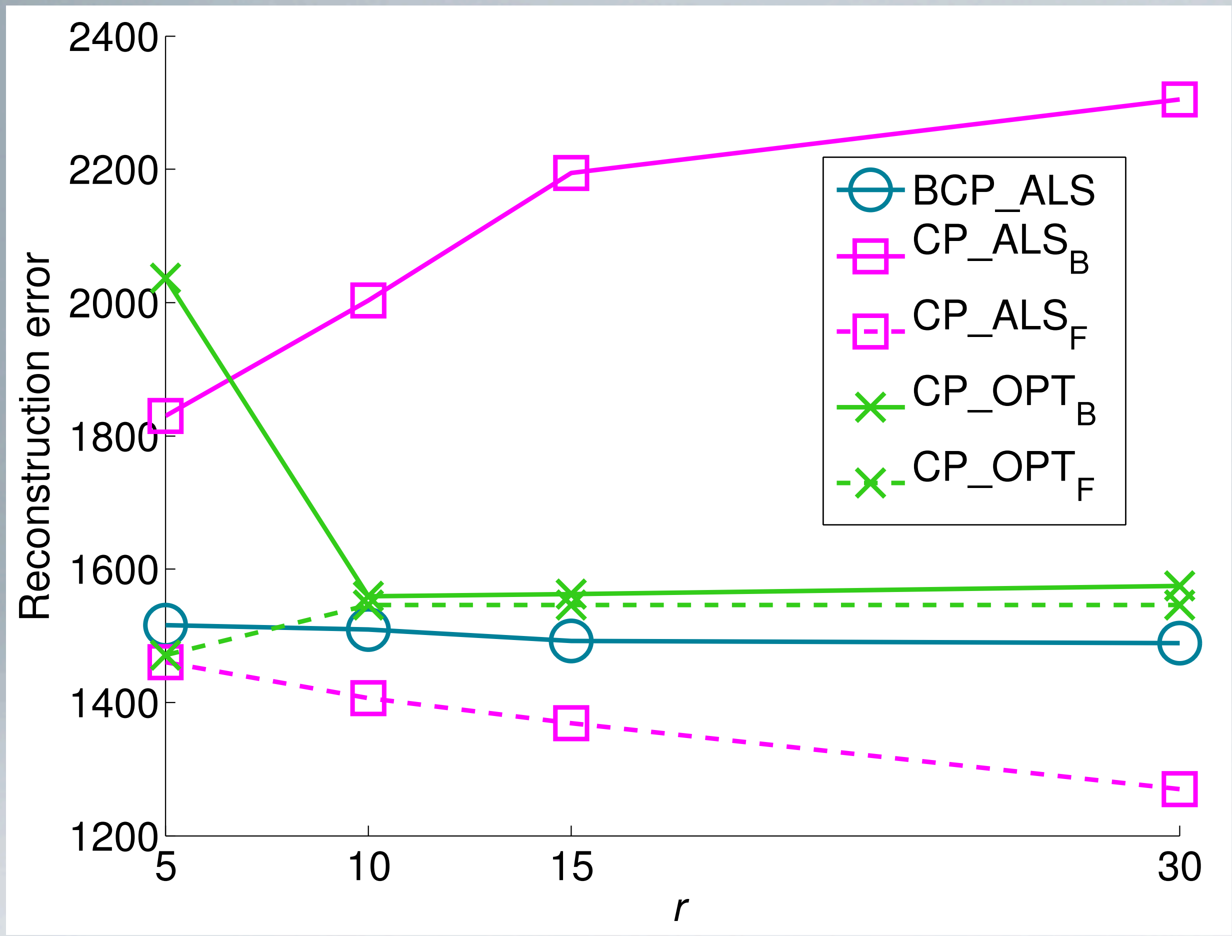
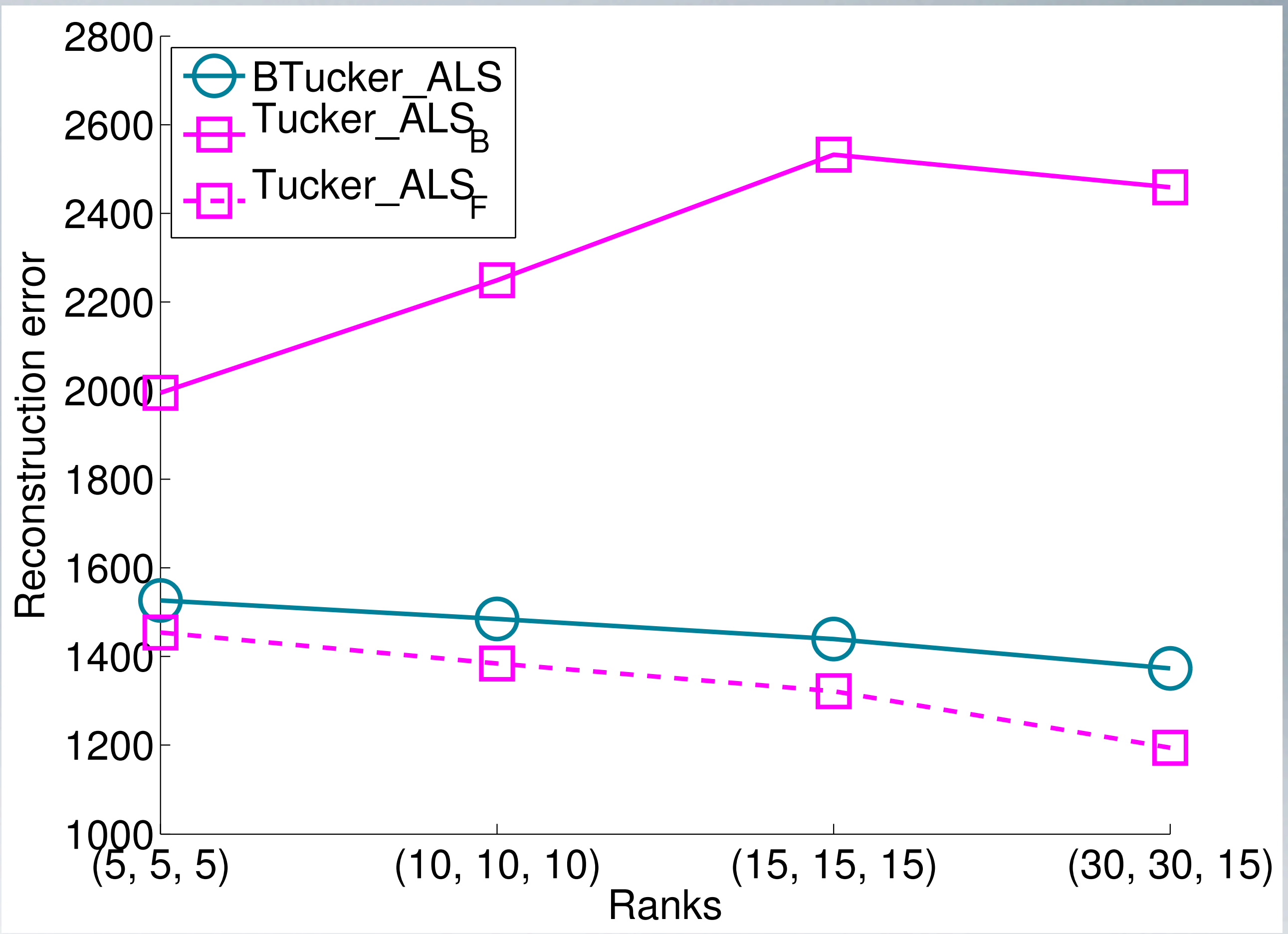




REAL-WORLD EXPERIMENTS





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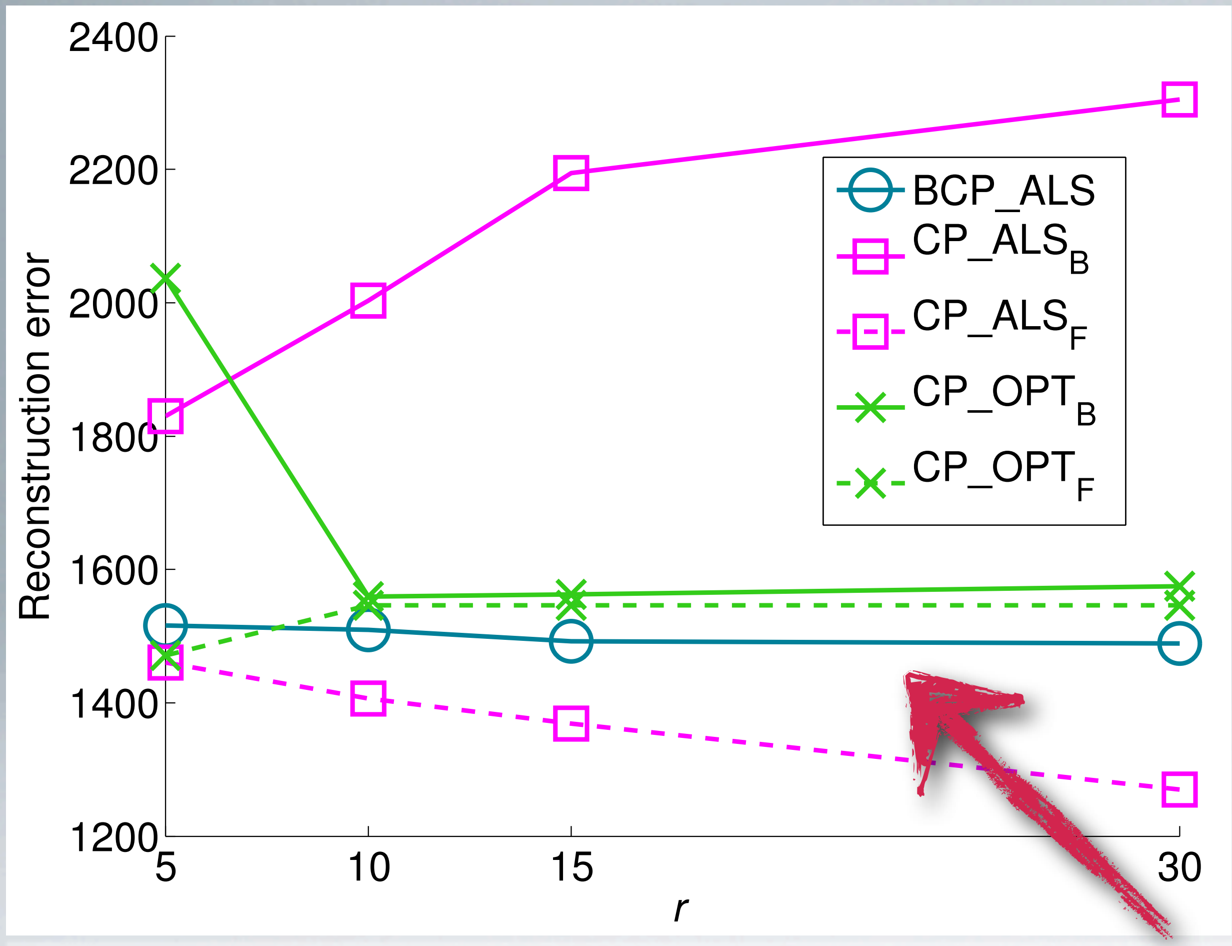

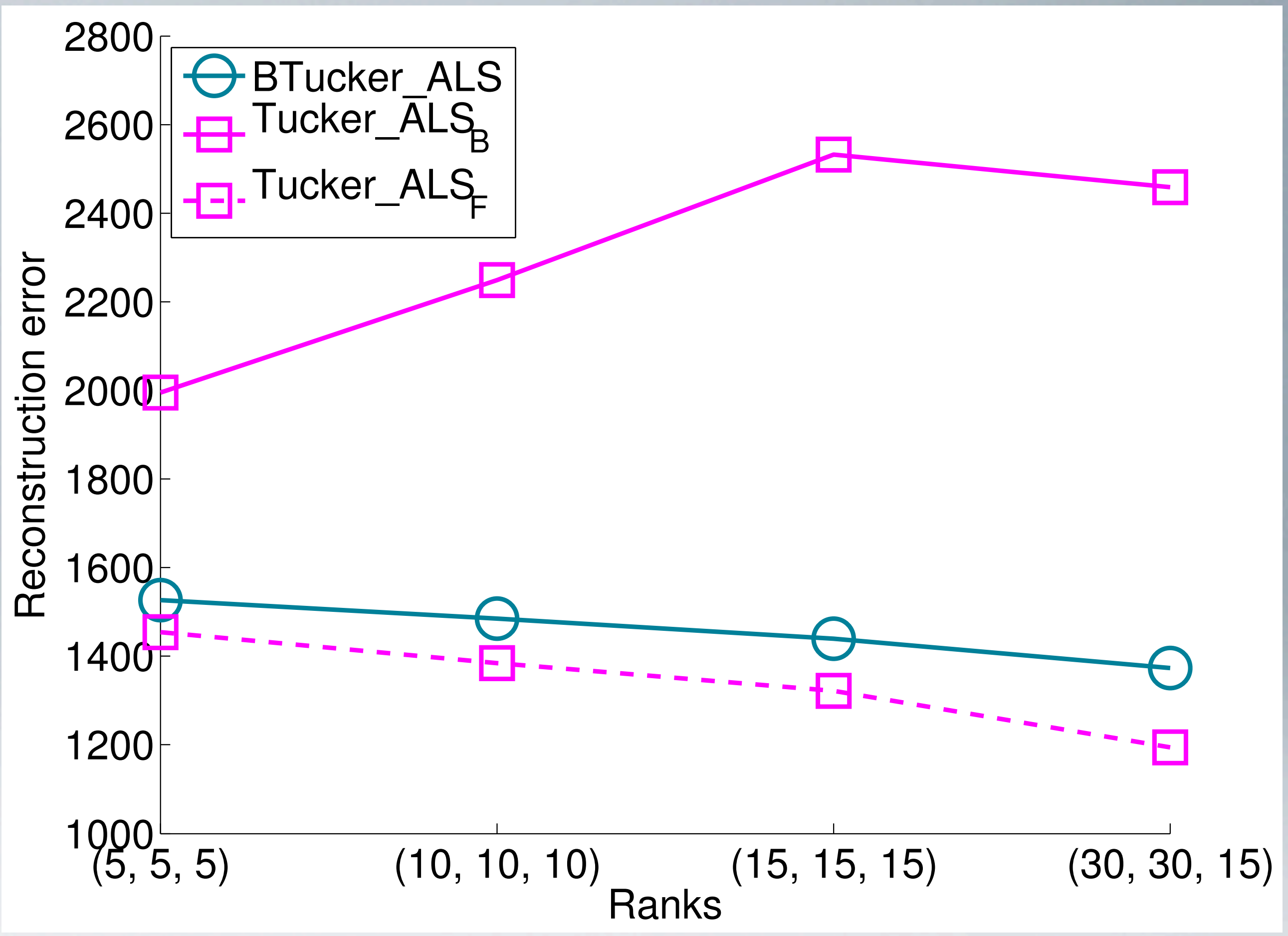

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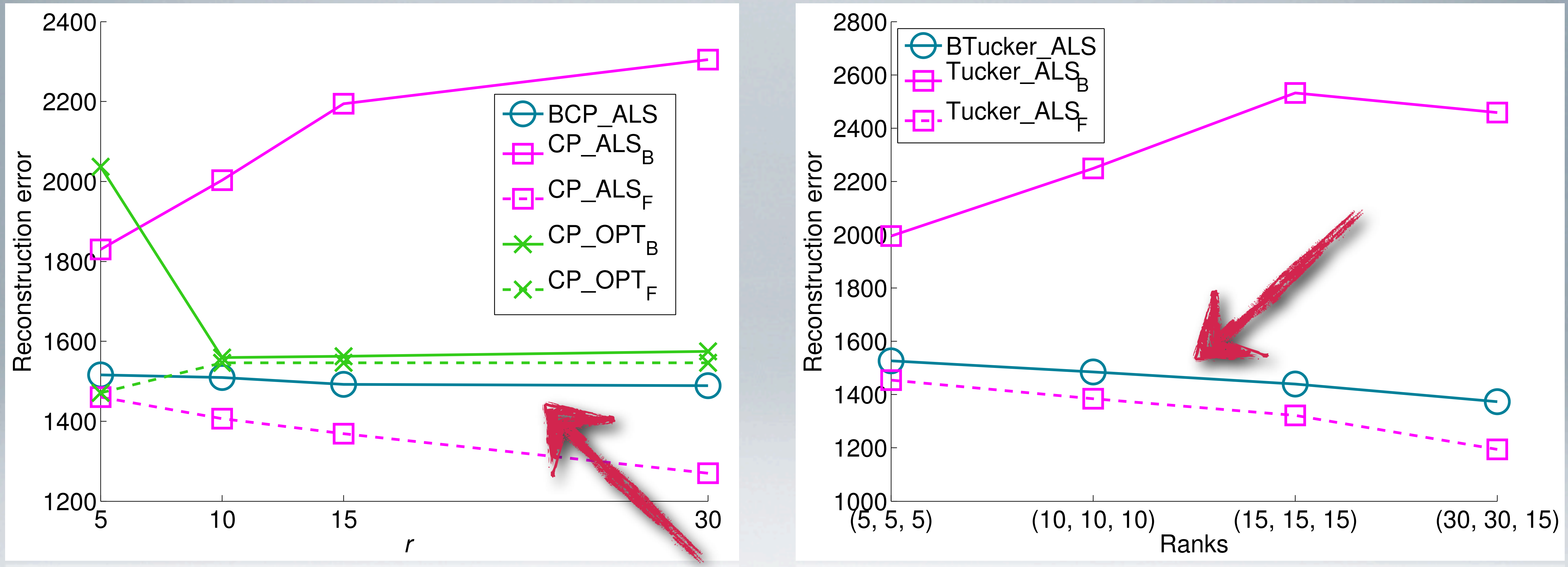

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  - And a bit like Boolean matrix factorizations
- They generalize other data mining techniques in many ways
- The playing field between Boolean and normal tensor factorizations is more level

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Thank You!